Monday Night Calculus, March 14, 2022 (π Day)

1. For $0 \le t \le 4$, a particle is moving along the x-axis. The particle's position is given by

$x(t) = 7t - 4t^2 + \int_0^t s^2\, ds$ (Michelle Slatcher, David Lederman)

(a) Find the position and velocity of the particle when $t = 3$.

$$x(3) = 7(3) - 4 \cdot 3^2 + \int_0^3 s^2\, ds$$

$$= 21 - 36 + \left[\frac{s^3}{3}\right]_0^3$$

$$= -15 + [9 - 0] = -6$$

$$v(t) = 7 - 8t + t^2$$

$$v(3) = 7 - 8 \cdot 3 + 3^2 = 7 - 24 + 9 = -8$$

(b) At $t = 3$ what is the speed of the particle? Is the speed increasing or decreasing? Give a reason for your answer.

$$\text{speed} = |v(3)| = |-8| = 8$$

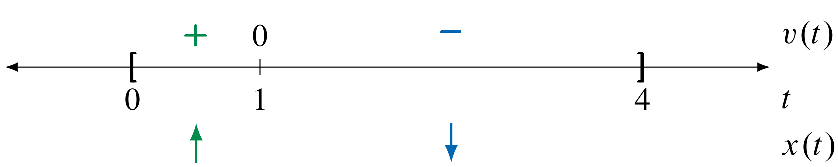
$$a(t) = -8 + 2t \quad \Rightarrow \quad a(3) = -8 + 6 = -2$$

$$v(3) = -8 < 0 \ \text{ and } \ a(3) = -2 < 0$$

Since velocity and acceleration have the same sign, the particle is speeding up at time $t = 3$.

(c) In the interval $(0, 4)$ the particle changes direction once. Find the value of t where the change of direction occurs.

$$v(t) = 0 : t^2 - 8t + 7 = (t - 1)(t - 7) = 0 \quad \Rightarrow \quad t = 1,\ 7$$



The particle changes direction at $t = 1$ because the velocity changes sign from positive to negative at that time.

(d) What is the value of t for which the particle is farthest right and at which it is farthest left. Justify your answer.

$$x(0) = 7 \cdot 0 - 4 \cdot 0^2 + \int_0^0 s^2\, ds = 0$$

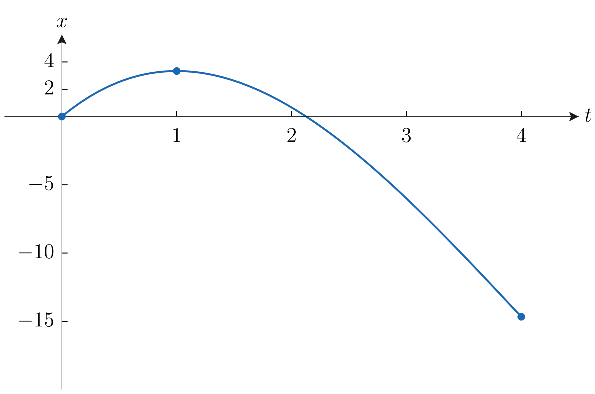
$$x(1) = 7 \cdot 1 - 4 \cdot 1^2 + \int_0^1 s^2\, ds = \frac{10}{3}$$

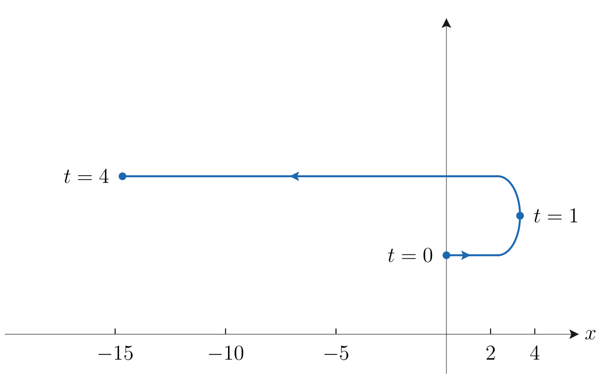
$$x(4) = 7 \cdot 4 - 4 \cdot 4^2 + \int_0^4 s^2\, ds = -\frac{44}{3}$$

t	$x(t)$
0	0
1	$\frac{10}{3}$
4	$-\frac{44}{3}$

The particle is farthest to the left when $t = 4$.

The particle is farthest to the right when $t = 1$.





2. The table below gives values of the differentiable function f and its derivative f' at selected values of x. (Stacey Katzer Oldfield)

x	2	5
$f(x)$	4	7
$f'(x)$	2	3

If $\int_2^5 f(x)\,dx = 14$, what is the value of $\int_2^5 x \cdot f'(x)\,dx$?

Solution

$$u = x \qquad dv = f'(x)\,dx$$

$$du = dx \qquad v = \int f'(x)\,dx = f(x)$$

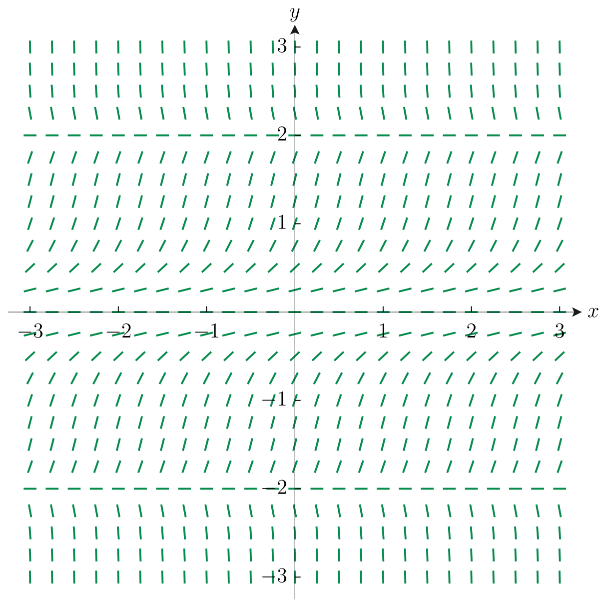
$$\int_2^5 x \cdot f'(x)\,dx = \left[x \cdot f(x) - \int f(x)\,dx\right]_2^5$$

$$= \left[x \cdot f(x)\right]_2^5 - \int_2^5 f(x)\,dx$$

$$= [5 \cdot f(5) - 2 \cdot f(2)] - 14$$

$$= 5 \cdot 7 - 2 \cdot 4 - 14 = 35 - 8 - 14 = 13$$

3. The slope field for the differential equation $\dfrac{dy}{dx} = y^2(4 - y^2)$ is shown below.



If $y = g(x)$ is the solution to the differential equation with the initial condition $g(-2) = -1$, then find $\lim\limits_{x\to\infty} g(x)$. (Nancy Smith)

Solution

$$\frac{dy}{dx} = y^2(4 - y^2)$$

$$\frac{dy}{y^2(2-y)(2+y)} = dx$$

$$\frac{1}{y^2(2-y)(2+y)} = \frac{A}{y} + \frac{B}{y^2} + \frac{C}{2-y} + \frac{D}{2+y}$$

$$1 = Ay(4-y^2) + B(4-y^2) + Cy^2(2+y) + Dy^2(2-y)$$

$$1 = 4Ay + (-B + 2C + 2D)y^2 + (-A + C - D)y^3 + 4B$$

$$\begin{cases} 4A & & & & = 0 \\ & -\; B & + 2C & + 2D & = 0 \\ -A & & + \; C & - \; D & = 0 \\ & 4B & & & = 1 \end{cases}$$

$$A = 0, \quad B = \frac{1}{4}, \quad C = \frac{1}{16}, \quad D = \frac{1}{16}$$

Back to the differential equation:

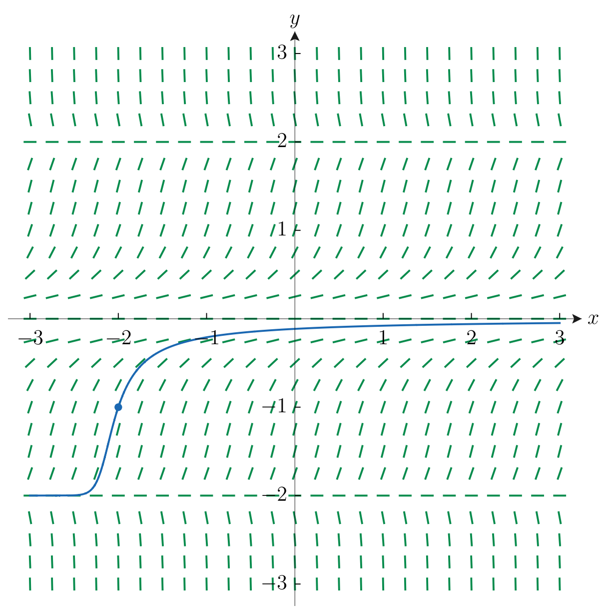
$$\left[\frac{1}{4}\frac{1}{y^2} + \frac{1}{16}\frac{1}{2-y} + \frac{1}{16}\frac{1}{2+y}\right] dy = dx$$

$$-\frac{1}{4y} - \frac{1}{16}\ln|2-y| + \frac{1}{16}\ln|2+y| = x + C$$

Use the initial condition: $g(-2) = -1$.

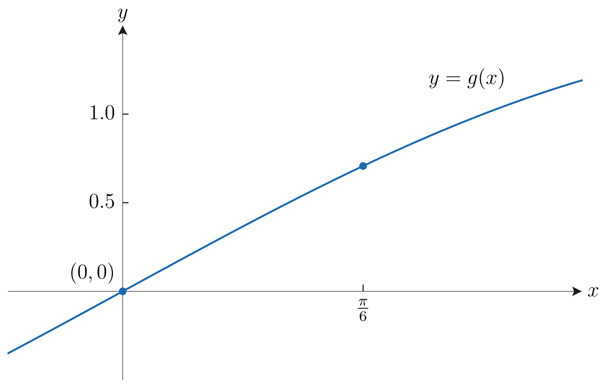
$$-\frac{1}{-4} - \frac{1}{16}\ln|3| + \frac{1}{16}\ln|1| = -2 + C \quad \Rightarrow \quad C = \frac{1}{16}(36 - \ln 3)$$

$$-\frac{1}{4y} - \frac{1}{16}\ln(2-y) + \frac{1}{16}\ln(2+y) = x + \frac{1}{16}(36 - \ln 3)$$



4. Let g be the function defined by $g(x) = \int_0^x \sqrt{\cos 2t}\,dt$. Find the length of the curve on the graph of g for $0 \le x \le \frac{\pi}{6}$

Solution



$$g'(x) = \sqrt{\cos 2x} \quad \Rightarrow \quad [g'(x)]^2 = \cos 2x$$

$$L = \int_0^{\pi/6} \sqrt{1 + \cos 2x}\,dx$$

$$= \int_0^{\pi/6} \sqrt{1 + (1 - 2\sin^2 x)}\,dx = \int_0^{\pi/6} \sqrt{2 - 2\sin^2 x}\,dx$$

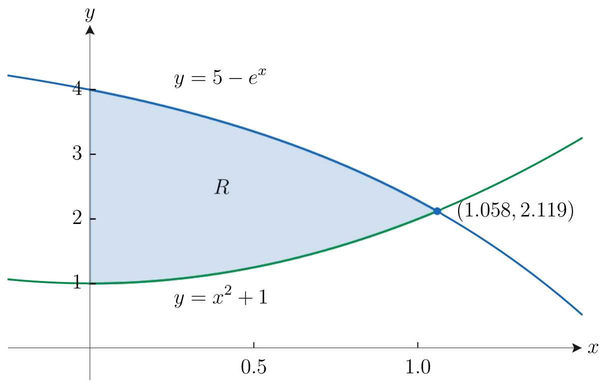
$$= \sqrt{2} \int_0^{\pi/6} \sqrt{1 - \sin^2 x}\,dx = \sqrt{2} \int_0^{\pi/6} \sqrt{\cos^2 x}\,dx$$

$$= \sqrt{2} \int_0^{\pi/6} \cos x\,dx = \sqrt{2} \Big[\sin x\Big]_0^{\pi/6}$$

$$= \sqrt{2}\left[\sin\frac{\pi}{6} - \sin 0\right] = \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}$$

5. Let R be the region in the first quadrant bounded by the y-axis and the graphs of $y = x^2 + 1$ and $y = 5 - e^x$.

(a) Find the area of the region R.

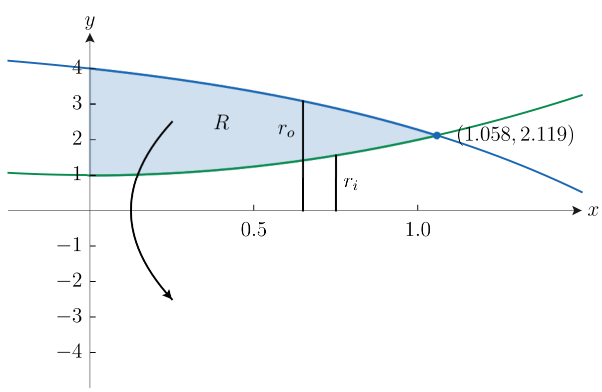


$$x^2 + 1 = 5 - e^x \quad \Rightarrow \quad x = 1.058 \qquad \text{Let } a = 1.058 \text{ and } b = f(a) = 2.119$$

$$A = \int_0^a [(5 - e^x) - (x^2 + 1)]\, dx = \int_0^a [4 - e^x - x^2]\, dx$$

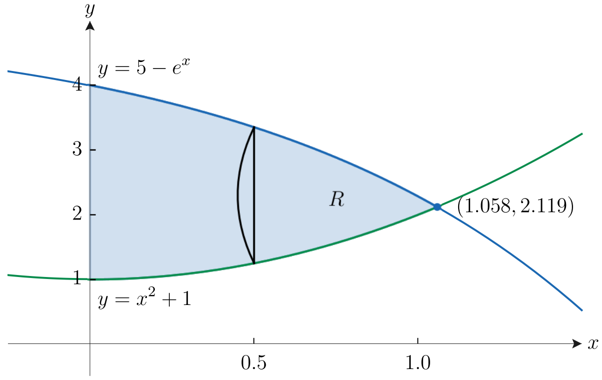
$$= \left[4x - e^x - \frac{x^3}{3}\right]_0^a = 1.957$$

(b) Find the volume of the solid that results when R is rotated about the x-axis.



$$V = \pi \int_0^a [(5 - e^x)^2 - (x^2 + 1)^2]\, dx = 28.841$$

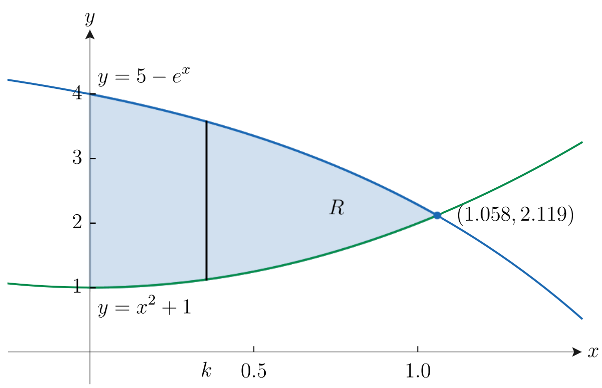
(c) The solid S has base R. Each cross-section perpendicular to the x-axis is a semicircle whose diameter lies in R. Find the volume of the solid S.



$$D = (5 - e^x) - (x^2 + 1) \quad \Rightarrow \quad r = (4 - e^x - x^2)/2$$

$$V = \int_0^a \pi \left[\frac{4 - e^x - x^2}{2}\right]^2 \cdot \frac{1}{2}\, dx = 1.736$$

(d) The vertical line $x = k$ divides the region R into two regions of equal areas. Write, but do not solve, an equation involving one or more integral expressions that could be used to determine the value of k.



$$\int_0^k [(5 - e^x) - (x^2 + 1)]\, dx = \int_k^a [(5 - e^x) - (x^2 + 1)]\, dx$$

Can we use technology to find k?